DEVIATION VECTORS IN COUPLED OSCILLATORS

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ABSTRACT

We study the behavior of the deviation vectors and of the corresponding spectra of stretching numbers in a system of weakly coupled oscillators, which is described by a four dimensional (4D) symplectic map composed of two 2D maps of the same form, plus a coupling term. The orbits we study are almost two dimensional, confined on a plane e.g. x_1x_2, while the other two components x_3, x_4 remain small. The deviation vectors of ordered orbits tend to fall on the x_1x_2 plane regardless of their initial orientation in the four dimensional phase space. The deviation vectors that are initially perpendicular to the x_1x_2 plane need a transient time to fall on the x_1x_2 plane. The spectral distance between the spectra derived by two different initial deviation vectors tends to a constant non-zero value if the orbit is ordered and to zero if the orbit is chaotic. The distinction is made faster if one vector is close to the x_1x_2 plane and the other perpendicular to this plane.

1. Introduction

The problem of distinguishing between ordered and chaotic motion in a dynamical system, especially one with many degrees of freedom, is fundamental in a large area of modern science. In recent years new techniques, that try to answer this problem, have been developed, based on the study of spectra of dynamical quantities. One of the most used quantities is the stretching number α:

$$\alpha = \ln\left|\frac{d(t+1)}{d(t)}\right| , \qquad (1)$$

where d(t) and d(t+1) are small deviations from a given orbit at successive times t and t+1 (Voglis and Contopoulos 1994, Contopoulos et al. 1995, Contopoulos and Voglis 1996, Voglis et al. 1999). The stretching number is a "short time Lyapunov characteristic number".

The basic tool in our study is the computation of the "spectral distance" D between two spectra $S_1(\alpha)$ and $S_2(\alpha)$ produced with different initial deviation vectors from the same orbit:

$$D^2 = \sum_{\alpha}[S_1(\alpha) - S_2(\alpha)]^2 \cdot \delta\alpha , \qquad (2)$$

where $\delta\alpha$ is the width of the bin of stretching numbers. If the motion is ordered the two spectra are different so D tends to a constant non-zero value, while if the motion is chaotic the final spectra are equal and D tends to zero (Voglis et al. 1999).

The dynamical system we study is the four dimensional (4D) symplectic map:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \cdot \sin(x_1 + x_2) - \mu \cdot [1 - \cos(x_1 + x_2 + x_3 + x_4)] \\ x_3' &= x_3 + x_4 \\ x_4' &= x_4 - \kappa \cdot \sin(x_3 + x_4) - \mu \cdot [1 - \cos(x_1 + x_2 + x_3 + x_4)] \end{aligned} \quad (\bmod 2\pi), \qquad (3)$$

composed of two 2-D maps of the same form:

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= x_2 - \nu \cdot \sin(x_1 + x_2) \end{aligned} \quad (\bmod 2\pi) \qquad (4)$$

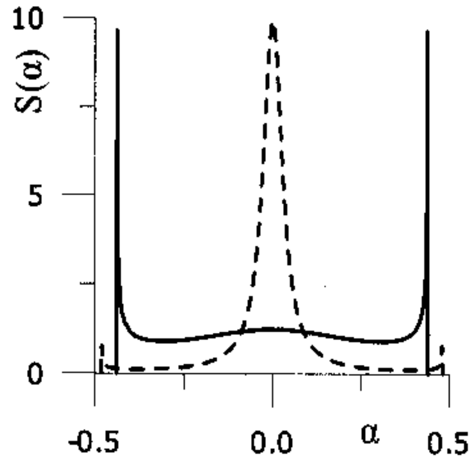

Figure 1. Spectra of stretching numbers for the ordered orbits of the 2D map (4), with initial conditions x_1=0.5, x_2=0 and initial deviation vector (1,1), after 10^6 iterations. The value of the parameter is $\nu=10^{-3}$ (dashed line) and $\nu=10^{-1}$ (solid line).

with parameters ν and κ, coupled with a term of order μ. This system is equivalent to a hamiltonian system with three degrees of freedom. All variables are given (mod 2π) so $x_i \in [-\pi,\pi)$, for i = 1,2,3,4. This map is a variant of Froeschlé's 4D symplectic map (1972). Contopoulos and Giorgilli (1988) studied the periodic orbits of this map and their bifurcations. Structures in the phase space of this map for small values of the coupling parameter μ were examined in detail by Skokos et al. (1997).

2. Dynamical spectra of ordered orbits

We study the spectra of stretching numbers of ordered orbits for small values of the coupling parameter μ. Our goal is to understand if and how the spectra of the two 2D maps (4) with parameters ν and κ, influence the spectrum in the 4D case when the 4D map (3) does not differ much from the case of two uncoupled 2D maps and the orbits are almost two dimensional. So we put $\nu=10^{-3}$, $\kappa=10^{-1}$ and $\mu=10^{-3}$. For the 2D map (4) the spectrum of stretching numbers inside the main island of stability around the stable periodic orbit $x_1=x_2=0$ does not change significantly for orbits with initial conditions near the origin. Since we study 4D orbits with initial conditions in these regions, the spectrum of the 4D case is the outcome of the combination of the spectra seen in figure 1.

In order to understand which 2D map, the one on plane x_1x_2 with $\nu=10^{-3}$, or the one on plane x_3x_4 with $\kappa=10^{-1}$, influence mostly the spectrum of the 4D map and also the role of the initial deviation vector, we compute in figure 2 the spectra of stretching numbers for some ordered orbits when the two 2D maps are uncoupled (μ=0, upper row) and weakly coupled ($\mu=10^{-3}$, lower row). In all cases we use the initial deviation vector (1,1,1,1) which has equal components on all projection planes and thus it is not particularly related to any of them. The other initial deviation vector has initially a component on one of the two planes x_1x_2 and x_3x_4 and particularly in only one axis. When $\mu=0$ the orbits we study are confined on one plane while the other two components remain always equal to zero since they are initially located on the origin of the other plane, which is a stable periodic orbit. For $\mu=10^{-3}$ the orbit is four dimensional but almost confined on the same plane as before because the two components that are initially located on the stable periodic orbit remain very close to it.

For $\mu=0$ the final spectrum we get from the vector (1,1,1,1) is almost identical to the one we get from the 2D map on the plane on which the 4D orbit is confined. So in figure 2a we have a spectrum (black line) similar to the dashed line of figure 1, since the orbit has non-zero components only on the x_1x_2 plane. When the non-zero components are on the x_3x_4 plane (figures 2b, c) we get a spectrum similar to the solid line of figure 1. The final spectrum depends on the initial deviation vector. When we have a vector with components initially on one of the two planes the final spectrum is similar to the one produced on this plane in the 2D case. So for the vector (0,0,1,0)

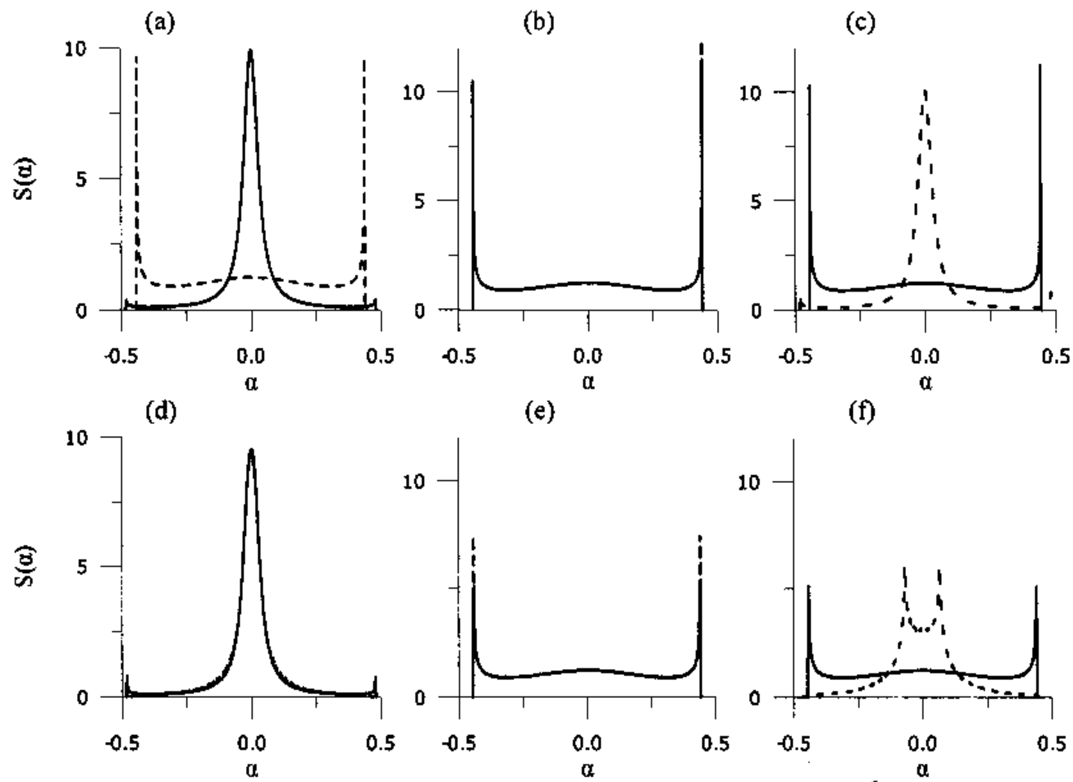

Figure 2. Spectra of stretching numbers of ordered orbits, after 10^6 iterations of the 4D map (3). The spectra in the upper row correspond to two uncoupled 2D maps ($\mu=0$) for various initial conditions and deviation vectors. The spectra in the lower row correspond to weekly coupled 2D maps ($\mu=10^{-3}$). The spectra in frames of the same column are produced from orbits with the same initial conditions and deviation vectors. In particular the initial conditions are $x_1=0.5$, $x_2=x_3=x_4=0$ (a, d) and $x_1=x_2=0$, $x_3=0.5$, $x_4=0$ (b, c, e, f) and the initial deviation vectors (1,1,1,1) (solid line), (0,0,1,0) (dashed line in a, b, d, e) and (1,0,0,0) (dashed line in c, f). The values of the parameters are $\nu=10^{-3}$ and $\kappa=10^{-1}$. In cases b, d and e the two spectra are very close thus practically only one curve is seen.

(figures 2a, b dashed line) we get the U-shaped spectrum seen in figure 1 (solid line) while the vector (1,0,0,0) (figure 2c dashed line) gives the dashed line of figure 1.

For $\mu=10^{-3}$ the spectrum produced by the vector (1,1,1,1) is similar to the one we had for $\mu=0$ (figures 2d, e, f solid lines). When the initial deviation vector has components only on one plane the spectrum is in general different from the one we had for $\mu=0$ (figures 2d, f dashed lines) and also it is different from the spectrum produced from the vector (1,1,1,1). We remark that even in figure 2d the two spectra are different since the one produced by the vector (0,0,1,0) does not have the two peaks at its edges. In figure 2e we see that the two spectra are almost identical because the vector (0,0,1,0) has its components on the plane x_3x_4 on which the orbit is confined, so it gives a spectrum similar to the one we get from the 2D map in the case of motion on the x_3x_4 plane (solid line of figure 1).

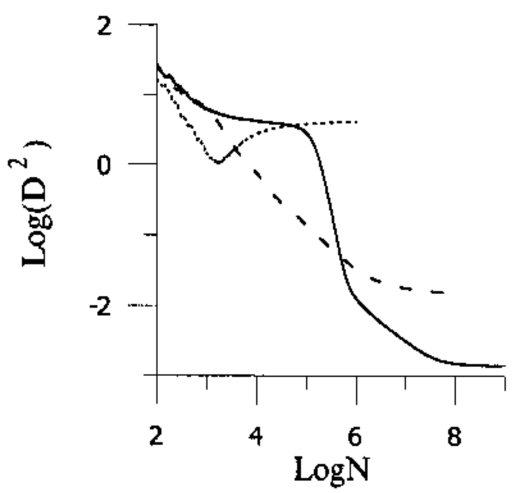

Figure 3. The square of spectral distance D between the spectra of figures 2d (solid line), 2e (dashed line) and 2f (dotted line) as function of the number of iterations N of the 4D map.

In figure 3 we see the evolution of the square of spectral distance D with respect to the number of iterations N for the cases of figures 2d, e and f. In all cases D^2 tends to a constant value. Even in the case of figure 2d, D tends to a small but non-zero value, since the two spectra are similar but not identical. In this case we see that up to 10^5 iterations D^2 forms a plateau, which means that the two spectra are significantly different. Then from 10^5 to 10^6 iterations D^2 decreases suddenly. The decrease continues in a smoother way up to 10^8 iterations and then D^2 seems to remain constant up to 10^9 iterations.

The spectra in figures 2e and f are produced by the same orbit. The two spectra in figure 2e are almost identical, thus D^2 decreases continuously and one has to wait up to 10^7 iterations in order to see that it tends to a non-zero value (figure 3 dashed line). If instead of using the vector (0,0,1,0) we use the vector (1,0,0,0) we can be sure that this orbit is ordered at least 100 times earlier, after about 10^5 iterations (figure 3 dotted line).

Now we investigate in detail the sudden decrease of D^2 seen in figure 3 (solid line). The spectrum produced by the vector (1,1,1,1) gets the form drawn in figure 2d with solid line, after a small number of iterations (less than 10^4) and does not change significantly up to 10^9 iterations. On the other hand the spectrum produced by the vector (0,0,1,0) changes its shape drastically mainly between 10^5 to 10^6 iterations. This behavior can be seen in figure 4.

In order to understand what causes this behavior we follow the evolution of the two initial deviation vectors (1,1,1,1) and (0,0,1,0). We remark that all the vectors are normalized having norm equal to 1. In figure 5 we plot the norm P_{12} of the projection of the normalized vectors on the x_1x_2 plane for 1,000 iterations in the beginning and after 10^5 and 10^6 iterations. We see that although initially (1,1,1,1) has equal components on both planes it stays mainly near the x_1x_2 plane in the beginning (figure 5a) and when the number of iterations increases it remains practically on the x_1x_2 plane (figures 5b, c, $P_{12} \cong 1$). On the other hand the behavior of the vectors produced by (0,0,1,0) is different. The vector is initially perpendicular to the x_1x_2 plane and stays almost exactly on the x_3x_4 plane in the beginning (figure 5a). Then we have a transient state where the vector goes away from the x_3x_4 plane and finally relaxes on the x_1x_2 plane. After 10^5 iterations the vector is really four dimensional since it has significant projections on both x_1x_2 and x_3x_4 planes (figure 5b). The corresponding spectrum after 10^5 iterations is seen in figure 4. After 10^6 iterations the vector produced from (0,0,1,0) has almost the same behavior as the vector produced by (1,1,1,1) after the first 1,000 iterations (figures 5a, c), thus it mostly lies on the x_1x_2 plane.

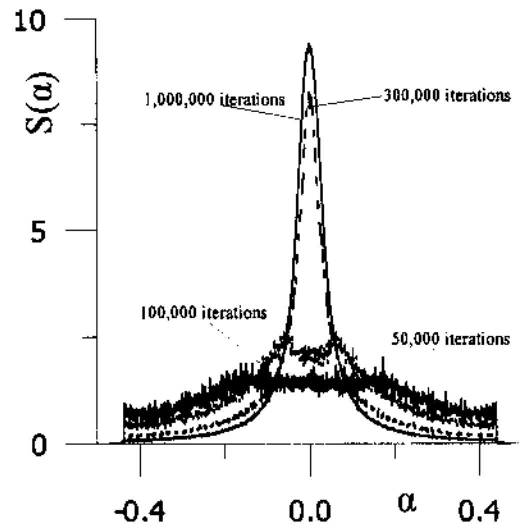

Figure 4. Evolution of the spectrum of stretching numbers of the orbit with initial conditions x_1=0.5, x_2=x_3=x_4=0 and initial deviation vector (0,0,1,0) after $5 \cdot 10^4$, 10^5, $3 \cdot 10^5$ and 10^6 iterations. The parameters of the 4D map are $\nu=10^{-3}$, $\kappa=10^{-1}$ and $\mu=10^{-3}$.

By examining many different initial deviation vectors we see that when the initial deviation vector has a component on the x_1x_2 plane, the spectrum we get is very similar to the one produced by the vector (1,1,1,1). Thus the spectral distance between them becomes very small and seems to tend to a constant value after more than 10^9 iterations. On the other hand the behavior of initial deviation vectors which are perpendicular to the x_1x_2 plane is similar to the behavior of the vector (0,0,1,0) i.e. the spectral distance D with respect to the spectrum produced by the vector (1,1,1,1), forms initially a plateau followed by a sudden jump to smaller values when the vector falls on the x_1x_2 plane. Thus in any case the deviation vector falls on the plane where the orbit is practically confined, although following different routes.

3. Conclusions

We studied four dimensional orbits that are almost confined on one plane e.g. x_1x_2 since the other two components, x_3 and x_4, remain always relatively small. Our results can be summarized as follows:

a) When a four dimensional orbit is practically confined near a two dimensional subspace of the phase space, the final form of the stretching numbers' spectrum is similar to the spectrum produced by a pure two dimensional orbit.

b) In the case of ordered orbits, the spectral distance D between the spectra produced by two different initial deviation vectors, finally becomes equal to a non-zero value. The number of iterations needed for this value to be reached depends on the choice of the initial deviation vectors.

c) The deviation vectors of the ordered orbits fall on the plane of motion regardless of their initial orientation in the four dimensional phase space.

d) If the initial deviation vector has components on the plane of motion falls on it after a small number of iterations. The distance between the spectra produced from these vectors stabilizes at a very small value after a rather large number of iterations.

e) If the initial deviation vector is perpendicular to the plane of motion it falls on it after many iterations. There exists a transient phase where the spectrum differs much from its final form. Thus the distance between the spectra produced by this initial deviation vector and one with components on the plane of motion is initially significant, but decreases rapidly when the vector falls on the plane of motion and then stabilizes faster than the case described in the previous paragraph. It is evident that this choice of initial deviation vectors helps us to decide the ordered nature of the orbit faster than any other choice we studied.

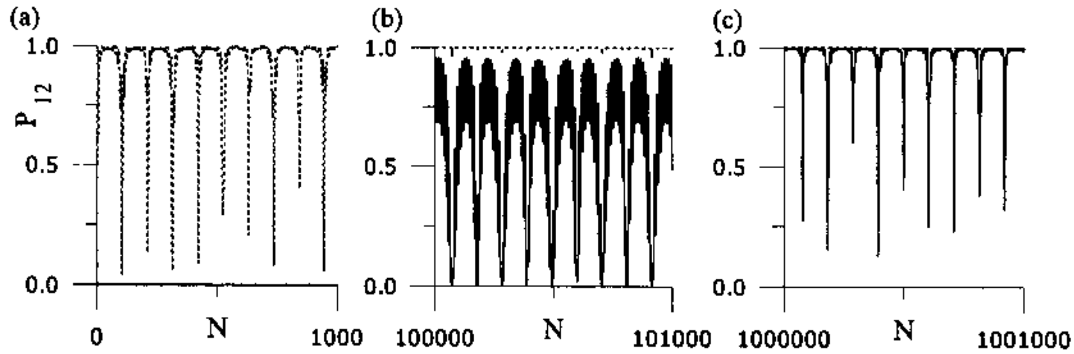

Figure 5. The norm P_{12} of the projection on the x_1x_2 plane of the normalized vectors produced by the vectors (1,1,1,1) (dotted line) and (0,0,1,0) (solid line) on the x_1x_2 plane with respect to the number of iterations of the 4D map. The two lines are not clearly seen in all the frames because the solid line in (a) practically coincides with the horizontal axis and the dotted lines in (b) and (c) with the line $P_{12}=1$. The orbit's initial conditions are $x_1=0.5$, $x_2=x_3=x_4=0$ and the parameters' values $\nu=10^{-3}$, $\kappa=10^{-1}$ and $\mu=10^{-3}$.

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